

CONTINUOUS FAMILIES OF RIEMANNIAN MANIFOLDS, ISOSPECTRAL ON FUNCTIONS BUT NOT ON 1-FORMS.

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ABSTRACT. The purpose of this paper is to present the first continuous families of Riemannian manifolds isospectral on functions but not on 1-forms, and simultaneously, the first continuous families of Riemannian manifolds with the same marked length spectrum but not the same 1-form spectrum. The examples presented here are Riemannian three-step nilmanifolds and thus provide a counterexample to the Ouyang-Pesce Conjecture for higher-step nilmanifolds. Ouyang and Pesce independently showed that all isospectral deformations of two-step nilmanifolds must arise from the Gordon-Wilson method for constructing isospectral nilmanifolds. In particular, all continuous families of Riemannian two-step nilmanifolds that are isospectral on functions must also be isospectral on p -forms for all p . They conjectured that all isospectral deformations of nilmanifolds must arise in this manner. These examples arise from a general method for constructing isospectral Riemannian nilmanifolds previously introduced by the author.

Section 1. Introduction.

The **spectrum** of a closed Riemannian manifold (M, g) , denoted $\text{spec}(M, g)$, is the collection of eigenvalues with multiplicities of the associated Laplace–Beltrami operator acting on smooth functions. Two Riemannian manifolds (M, g) and (M', g') are said to be **isospectral** if $\text{spec}(M, g) = \text{spec}(M', g')$. The Laplace–Beltrami operator may be extended to act on smooth p -forms by $\Delta = d\delta + \delta d$, where δ is the metric adjoint of d . We denote its eigenvalue spectrum by $p\text{-spec}(M, g)$.

This paper addresses the question of spectral rigidity: “If we continuously deform a manifold while keeping the Laplace spectrum constant, must the resulting manifolds be isometric?” That is, can we determine the geometry of a manifold based on nearby behavior of the spectrum?

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Sometimes the answer is “Yes.” Croke and Sharafutdinov [CS], generalizing work of Guillemin and Khazdan [GuK] for surfaces, have recently showed spectral rigidity of negatively curved manifolds. Flat tori are known to be spectrally rigid [Wp], while Heisenberg manifolds [P1] are spectrally rigid within the family of left invariant metrics.

Nilmanifolds and nilpotent Lie groups have played a vital role in showing that the general answer is “No.” In 1984, Gordon and Wilson [GW1] constructed the first nontrivial isospectral deformations, continuous families of isospectral, nonisometric Riemannian nilmanifolds. The Gordon–Wilson method has been modified to produce conformally equivalent isospectral deformations of Riemannian two-step nilspaces [DG2], [BG], and nontrivial isospectral deformations arbitrarily close to Heisenberg manifolds [Sch2]. Using new techniques, Gordon and Wilson [GW3] have recently used nilmanifolds to construct continuous families of isospectral (Dirichlet and Neumann) manifolds with boundary that are nonlocally isometric. These are the only known examples of nontrivial isospectral deformations.

While two-step nilmanifolds are not spectrally rigid, they do satisfy a “modified” rigidity. In 1992, Ouyang [O] and Pesce [P3], independently showed that isospectral deformations of two-step nilmanifolds must arise from the Gordon–Wilson method. Ouyang and Pesce conjectured, “*isospectral deformations of nilmanifolds must arise from the Gordon–Wilson method.*” It was expected that a combination of their different techniques would prove the result on higher-step nilmanifolds.

In 1985, Sunada [Su] gave an elegant method for constructing pairs of isospectral manifolds that can be explained solely in representation theoretic terms. (See [B2] for a discussion.) His method was later generalized [DG2] to include the Gordon–Wilson deformations. Sunada’s method and its generalizations produce manifolds that are *strongly* isospectral, that is, they must share the same spectrum for all natural, strongly elliptic operators. In particular, isospectral deformations arising from the Gordon–Wilson method must share the same p -form spectrum for all p . Conversely, Schueth [Sch1] showed that if one uses representation theoretic techniques alone to obtain continuous families of isospectral nilmanifolds, then the deformation must arise from the Gordon–Wilson method.

As most constructions for producing families of isospectral manifolds can be explained by Sunada’s method or its generalizations, examples of Riemannian manifolds that are isospectral on functions but not on forms are sparse. Gordon [G2] has constructed pairs of Heisenberg manifolds that are isospectral on functions but not on 1-forms, and the author [Gt3,4] has constructed pairs of three-step Riemannian nilmanifolds with this property. The only other known examples of manifolds that are isospectral on functions but not on p -forms for all p are lens spaces. For any choice of $P \in \mathbb{Z}^+$, Ikeda [I2] has constructed examples of pairs isospectral lens spaces that are isospectral on p -forms for

$p = 0, 1, \dots, P$ but not isospectral on $(P + 1)$ -forms. Recently, Pesce [P4] has been able to explain the Ikeda examples in a Sunada-like setting, which requires a genericity assumption excluding nilmanifolds. Nilmanifolds, therefore, now account for all known isospectral manifolds that do not fall under a Sunada-like setting.

The main result of this paper is the following.

Theorem: There exist continuous families of Riemannian manifolds that are isospectral on functions but not on 1-forms.

The examples presented here are three-step nilmanifolds, and thus provide a counterexample to the Ouyang–Pesce conjecture. The new deformations arise from a general method for constructing isospectral Riemannian nilmanifolds introduced in [Gt4], which combines techniques from representation theory and Riemannian geometry.

All known examples of strongly isospectral nilmanifolds arise from the Gordon–Wilson method, leading to the following.

Conjecture: Continuous families of nilmanifolds that are isospectral on functions and on 1-forms must arise from the Gordon–Wilson method.

Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) have the same **marked length spectrum** if there exists an isomorphism between the fundamental groups of M_1 and M_2 such that corresponding free homotopy classes of loops contain smoothly closed geodesics of the same length. (See Section 5 for more details.)

The relationship between the Laplace spectrum and lengths of closed geodesics arises from the study of the wave equation (see [DGu]), and in the case of compact, hyperbolic manifolds, from the Selberg Trace Formula (see [C, Chapter XI]). Colin de Verdière [CdV] has shown that generically, the Laplace spectrum determines the lengths of closed geodesics. All known examples of manifolds that are isospectral have the same lengths of closed geodesics.

Croke [Cr] and Otal [Ot12] independently showed that if a pair of compact surfaces with negative curvature have the same marked length spectrum, they are necessarily isometric. This is due to the fact that in certain cases, the marked length spectrum and the geodesic flow are (roughly) equivalent notions. Gordon, Mao, and Schueth [GM1,2], [GMS] have shown that generically, two-step nilmanifolds with conjugate geodesic flows must be isometric.

Isospectral Riemannian manifolds constructed using the Gordon–Wilson method necessarily have the same marked length spectrum, while Eberlein [E] proved the converse in the case of two-step nilmanifolds. Thus the Gordon–Wilson method produces continuous families of strongly isospectral Riemannian manifolds with the same marked length spectrum. In contrast, we prove the following.

Theorem: There exist continuous families of Riemannian manifolds

with the same marked length spectrum but not the same spectrum on 1-forms.

Discrete families of manifolds with this property were presented in [Gt4], to which the reader is referred for an introduction to the relationships among the marked length spectrum, the length spectrum, the Laplace spectrum on functions, and the Laplace spectrum on forms on discrete families of Riemannian manifolds. The only other known examples of manifolds that have the same marked length spectrum but not the same spectrum on 1-forms are the Zoll spheres and standard spheres, which are not even isospectral on functions.

This paper is organized as follows. We briefly review background material and the Gordon–Wilson method in Section 2. The techniques used to construct the new examples are discussed in Section 3. One way to show that isospectral nilmanifolds cannot arise from the Gordon–Wilson method is by showing that no diffeomorphism between the manifolds of the correct type can exist. This approach is discussed in Section 3, as it lends insight into the types of Riemannian nilmanifolds that can support isospectral deformations that are not strongly isospectral. The new examples are presented in Section 4, while the marked length spectrum and the 1-form spectrum are the focus of Sections 5 and 6, respectively.

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Section 2. Background and Notation.

2.1 Riemannian Nilmanifolds. Let G be a simply connected Lie group, and let Γ be a cocompact, discrete subgroup of G . A Riemannian metric g is **left invariant** if the left translations of G are isometries. The left invariant metric g descends to a Riemannian metric on $\Gamma \backslash G$, also denoted by g . Note that a left invariant metric is determined by specifying an orthonormal basis $\{E_1, \dots, E_n\}$ of the Lie algebra \mathfrak{g} of G . As G is unimodular, the Laplace–Beltrami operator Δ of $(\Gamma \backslash G, g)$ is just

$$\Delta = - \sum_{i=1}^n E_i^2.$$

For a Lie algebra \mathfrak{g} , denote by $\mathfrak{g}^{(1)}$ the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ of \mathfrak{g} . Inductively, define $\mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}]$. A Lie algebra \mathfrak{g} is **k -step nilpotent** if $\mathfrak{g}^{(k-1)}$ is nontrivial and central. A Lie group G is k -step nilpotent if its Lie algebra is. Let Γ be a cocompact, discrete subgroup of a simply connected nilpotent Lie group G with left invariant metric g . The locally homogeneous space $(\Gamma \backslash G, g)$ is called a **Riemannian nilmanifold**.

For more details about nilmanifolds and nilpotent Lie groups see [CG].

2.2 Isometries of Nilmanifolds. We write $(M, g) = (M', g')$ if the Riemannian manifolds (M, g) and (M', g') , are isometric. Let $(\Gamma \backslash G, g)$ be a Riemannian nilmanifold. Any automorphism $\Phi \in \text{Aut}(G)$ induces an isometry from $(\Gamma \backslash G, \Phi^*g)$ to $(\Phi(\Gamma) \backslash G, g)$. Any inner automorphism $I_x \in \text{IA}(G)$ induces an isometry from $(\Gamma \backslash G, g)$ to $(\Gamma \backslash G, I_x^*g)$ via the mapping $R_x : \Gamma \backslash G \rightarrow \Gamma \backslash G$ defined by $R_x(\Gamma y) = \Gamma yx$. Let $\text{OA}(G, g) = \{\Upsilon \in \text{Aut}(G) : \Upsilon^*g = g\}$, the orthogonal automorphisms of G with respect to g . If Υ is in $\text{OA}(G, g)$, then $(\Gamma \backslash G, g) = (\Gamma \backslash G, \Upsilon^*g)$.

Recall that for any Lie group G with Lie algebra \mathfrak{g} , $\text{Aut}(\mathfrak{g})$ is a Lie group with Lie algebra $\text{Der}(\mathfrak{g})$. We denote by $\exp : \text{Der}(\mathfrak{g}) \rightarrow \text{Aut}(\mathfrak{g})$ the Lie algebra exponential, which reduces to the matrix exponential by looking at matrix representations.

2.3 Notation. For every $D \in \text{Der}(\mathfrak{g})$, we associate an operator D^* that takes the two-form ω to the two-form defined by $D^*\omega(X, Y) = \omega(DX, Y) + \omega(X, DY)$.

One generally denotes by $\Phi^*\omega$ the two-form defined by $\Phi^*\omega(X, Y) = \omega(\Phi_*(X), \Phi_*(Y))$, where $\Phi_* \in \text{Aut}(\mathfrak{g})$. As $D \in \text{Der}(\mathfrak{g})$ is the derivative of a curve Φ_{s*} in $\text{Aut}(\mathfrak{g})$, $D^*\omega$ is the derivative of $\Phi_s^*\omega$ in the obvious sense.

With this notation, $\text{OA}(G, g)$ is a Lie group with Lie algebra $\text{OD}(G, g) = \{D \in \text{Der}(\mathfrak{g}) : D^*g = 0\}$, the skew-symmetric derivations of \mathfrak{g} .

2.4 Almost Inner Automorphisms [GW1], [G1]. Let G be a Lie group with Lie algebra \mathfrak{g} and cocompact, discrete subgroup Γ . A Lie group automorphism Ψ of G is **almost inner** if for all x in G , the elements x and $\Psi(x)$ are conjugate in G . The automorphism Ψ of G is Γ -**almost inner** if for all γ in Γ , the elements γ and $\Psi(\gamma)$ are conjugate in G . A Lie algebra isomorphism Ψ_* is **almost inner** (resp., Γ -**almost inner**) if for every X in \mathfrak{g} , (resp., X in $\log \Gamma$), $\Psi_*(X) = \text{Ad}(x)(X)$ for some x in \mathfrak{g} . Here Ad denotes the adjoint representation of G on \mathfrak{g} . A derivation D is inner if there exists Y in \mathfrak{g} such that $D = \text{ad}(Y)$. A derivation D is **almost inner**, (resp., Γ -**almost inner**) if for every X in \mathfrak{g} (resp., X in $\log \Gamma$) there exists Y in \mathfrak{g} such that $D(X) = \text{ad}(Y)(X)$.

Denote by $\text{IA}(G)$, (resp., $\text{AIA}(G)$, $\text{AIA}(G; \Gamma)$) the set of inner (resp., almost inner, Γ -almost inner) automorphisms of G . Note that $\text{IA}(G) \subset \text{AIA}(G) \subset \text{AIA}(G; \Gamma)$, and there exist examples [Gt1], [GW1] demonstrating that each of these containments can be strict. Define $\text{IA}(\mathfrak{g})$, $\text{AIA}(\mathfrak{g})$, $\text{AIA}(\mathfrak{g}; \Gamma)$, $\text{ID}(\mathfrak{g})$, $\text{AID}(\mathfrak{g})$, and $\text{AID}(\mathfrak{g}; \Gamma)$ analogously.

If G is nilpotent, then $\Psi \in \text{Aut}(G)$ is inner (resp., almost inner, Γ -almost inner) if and only if $\Psi_* \in \text{Aut}(\mathfrak{g})$ is inner (resp., almost inner, Γ -almost inner). Also, $\text{IA}(\mathfrak{g})$, $\text{AIA}(\mathfrak{g})$, and $\text{AIA}(\mathfrak{g}; \Gamma)$ are connected Lie subgroups of $\text{Aut}(\mathfrak{g})$, with Lie algebras $\text{ID}(\mathfrak{g})$, $\text{AID}(\mathfrak{g})$, and $\text{AID}(\mathfrak{g}; \Gamma)$, respectively.

2.5 Theorem [GW1], [G2]. *Let Γ be a cocompact, discrete subgroup of a*

simply connected, nilpotent Lie group G with left invariant metric g .

- (1) If Ψ is a Γ -almost inner automorphism, then $p\text{-spec}(\Gamma \backslash G, g) = p\text{-spec}(\Psi(\Gamma) \backslash G, g)$ for $p = 0, 1, \dots, \dim(G)$.
- (2) If Ψ_s is a continuous family of Γ -almost inner automorphisms that are not inner automorphisms, then $(\Psi_s(\Gamma) \backslash G, g) = (\Gamma \backslash G, \Psi_s^* g)$ is a nontrivial, isospectral deformation.
- (3) There exist two-step nilpotent Lie groups supporting continuous families of Γ -almost inner automorphisms that are not inner automorphisms.

Conversely, Ouyang and Pesce independently proved the following.

2.6 Theorem [O],[OP],[P2]. *Let $(\Gamma \backslash G, g_s)$ be a continuous family of Riemannian two-step nilmanifolds that are isospectral on functions. Then there exists a continuous family Ψ_s of Γ -almost inner automorphisms with $\Psi_0 = I$ such that $g_s = \Psi_s^* g_0$.*

2.7 Notation and Remarks. Define

$$\text{GW}(G, g; \Gamma) = \text{OA}(G, g) \circ \text{AIA}(G; \Gamma) = \{\Upsilon \circ \Psi : \Upsilon \in \text{OA}(G, g), \Psi \in \text{AIA}(G; \Gamma)\},$$

$$\text{GW}(G, g) = \text{OA}(G, g) \circ \text{AIA}(G) = \{\Upsilon \circ \Psi : \Upsilon \in \text{OA}(G, g), \Psi \in \text{AIA}(G)\}.$$

The group $\text{AIA}(G)$ is a normal subgroup of $\text{Aut}(G)$, so $\text{GW}(G, g)$ is a Lie group with Lie algebra $\text{OD}(\mathfrak{g}, g) \oplus \text{AID}(\mathfrak{g})$. In particular, if $D \in \text{OD}(\mathfrak{g}, g) \oplus \text{AID}(\mathfrak{g})$, then the Lie group automorphism Φ defined at the Lie algebra level by $\Phi_* = \exp(D)$ is in $\text{GW}(G, g)$. A priori, the set $\text{GW}(G, g; \Gamma)$ is not a subgroup of $\text{Aut}(G)$, as there is no reason to assume that $\text{AIA}(G; \Gamma)$ is a normal subgroup of $\text{Aut}(G)$, although this proves to be the case in all known examples.

For the nilpotent Lie groups studied in Section 4, $\text{AIA}(G) = \text{AIA}(G; \Gamma)$ for any choice of cocompact, discrete subgroup Γ of G .

2.8 Proposition [GW1]. *Let G be a simply connected, nilpotent Lie group with cocompact discrete subgroup Γ and left invariant metric g . Let Φ_s be a continuous family of automorphisms of G such that Φ_0 is the identity.*

- (1) *The continuous family $(\Gamma \backslash G, \Phi_s^* g)$ is trivial if and only if $\Phi_s \in \text{OA}(G, g) \circ \text{IA}(G)$ for all s .*
- (2) *The continuous family $(\Gamma \backslash G, \Phi_s^* g)$ is Gordon–Wilson if and only if there exists Υ in $\text{OA}(G, g)$ such that for all s , $\Phi_s \circ \Upsilon \in \text{GW}(G, g; \Gamma)$.*

Note that by (2.7), if $\text{GW}(G; \Gamma)$ is a group, then we may assume $\Upsilon = I$ in the statement of Proposition 2.8.

Comments on Proof. This follows directly from [GW1, Corollary 5.3], using $\mathcal{D} = \{\delta \in \text{Aut}(G) : \delta(\Gamma) = \Gamma\}$ discrete and Φ_0 the identity. Note that the group defined by Gordon–Wilson as K_g is denoted here by $\text{OA}(G, g)$. \square

2.9 Quotient Nilmanifolds. Let G be a simply connected, k -step nilpotent Lie group with Lie algebra \mathfrak{g} . Define \bar{G} to be the simply connected, $(k-1)$ -step nilpotent Lie group $\bar{G} = G/G^{(k-1)}$. For Γ a cocompact, discrete subgroup of G , denote by $\bar{\Gamma}$ the image of Γ under the canonical projection from G onto \bar{G} . The group $\bar{\Gamma}$ is then a cocompact, discrete subgroup of \bar{G} . We denote elements of \bar{G} by \bar{x} , where \bar{x} is the image of x under the canonical projection from G onto \bar{G} . Note that $\bar{x} = \bar{y}$ if and only if xy^{-1} is in $G^{(k-1)} \subset Z(G)$. If $\Phi \in \text{Aut}(G)$, we denote by $\bar{\Phi} \in \text{Aut}(\bar{G})$ the projection of Φ onto \bar{G} . The Lie algebra $\bar{\mathfrak{g}}$ of \bar{G} is just $\mathfrak{g}/\mathfrak{g}^{(k-1)}$. We denote elements of $\bar{\mathfrak{g}}$ by \bar{X} , where \bar{X} is the image of X under the canonical projection from \mathfrak{g} onto $\bar{\mathfrak{g}}$.

For a left invariant metric g on G , we associate a left invariant metric \bar{g} on \bar{G} by restricting g to an orthogonal complement of $\mathfrak{g}^{(k-1)}$ in \mathfrak{g} . We call the $(k-1)$ -step nilmanifold $(\bar{\Gamma} \backslash \bar{G}, \bar{g})$ the **quotient nilmanifold** of $(\Gamma \backslash G, g)$. Using the definition of \bar{g} , one easily sees that the projection $(\Gamma \backslash G, g) \rightarrow (\bar{\Gamma} \backslash \bar{G}, \bar{g})$ is a Riemannian submersion with totally geodesic fibers.

2.10 Definition. The simply connected nilpotent Lie group G is **strictly nonsingular** if for every noncentral x in G , if $\bar{x} = \bar{v}$ then x and v are conjugate in G . Restated, for every noncentral x in G and every z in $Z(G)$, there exists y in G such that $xyx^{-1} = xz$. The nilpotent Lie algebra \mathfrak{g} is **strictly nonsingular** if for every noncentral X in \mathfrak{g} and every Z in \mathfrak{z} there exists Y in \mathfrak{g} such that $[X, Y] = Z$. That is, for every noncentral X in \mathfrak{g} , $\mathfrak{z} \subset \text{ad}(X)(\mathfrak{g})$.

2.11 Remarks.

- (1) One easily checks that these notions are equivalent, i.e., a nilpotent Lie group is strictly nonsingular if and only if its Lie algebra is strictly nonsingular.
- (2) Equivalently, a nilpotent Lie group G is strictly nonsingular if and only if every fiber of the submersion $G \rightarrow \bar{G}$ is contained in a single orbit of the action of G on itself by conjugation. A two-step nilpotent Lie group is strictly nonsingular if and only if every fiber is equal to an orbit.
- (3) If G is strictly nonsingular, then $Z(G) = G^{(k-1)}$, but not conversely.
- (4) The examples presented in Section 4 show that one can have both G and \bar{G} strictly nonsingular, but this is not the case in general. See [Gt1,2] for examples.

Section 3. New Deformations.

The method we will use to construct the new isospectral deformations is the following. This method has previously produced *discrete* families of Riemannian manifolds, isospectral on functions but not on 1-forms [G2], [Gt3,4].

3.1 Theorem [Gt2, Theorem 3.3]. *Let G be a simply connected, strictly non-singular nilpotent Lie group with left invariant metric g . If Γ_1 and Γ_2 are*

cocompact, discrete subgroups of G such that

$$\Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G) \quad \text{and} \quad \text{spec}(\bar{\Gamma}_1 \backslash \bar{G}, \bar{g}) = \text{spec}(\bar{\Gamma}_2 \backslash \bar{G}, \bar{g}),$$

then $\text{spec}(\Gamma_1 \backslash G, g) = \text{spec}(\Gamma_2 \backslash G, g)$.

3.2 Remark. Any deformation using Theorem 3.1 must be of the form $(\Gamma_s \backslash G, g)$, where Γ_s depends continuously on s . In particular, the manifolds must have a common Riemannian covering (G, g) , hence are locally isometric. It is known [Sch1] that if Γ_s is any continuous family of cocompact, discrete subgroups of a nilpotent Lie group G , then there exists a continuous family of automorphisms $\Phi_s \in \text{Aut}(G)$ with $\Phi_0 = I$ such that $\Gamma_s = \Phi_s(\Gamma)$. So the deformations arising from Theorem 3.1 are of the form $(\Gamma \backslash G, \Phi_s^*g)$, $\Phi_s \in \text{Aut}(G)$.

3.3 Corollary. Let G be a strictly nonsingular nilpotent Lie group with left invariant metric g and cocompact, discrete subgroup Γ . Assume there exists a continuous family of automorphisms $\Phi_s \in \text{Aut}(G)$, $\Phi_0 = I$, such that for all s

- (1) Φ_s restricted to $Z(G)$ is the identity, and
- (2) $\bar{\Phi}_s \in \text{GW}(\bar{G}, \bar{g}; \bar{\Gamma})$.

Then $(\Gamma \backslash G, \Phi_s^*g)$ is an isospectral deformation. The deformation is trivial if and only if $\Phi_s \in \text{OA}(G, g) \circ \text{IA}(G)$ for all s . The deformation is Gordon–Wilson if and only if there exists $\Upsilon \in \text{OA}(G, g)$ such that $\Phi_s \circ \Upsilon \in \text{GW}(G, g; \Gamma)$ for all s .

Note again that by (2.7), if $\text{GW}(G, g; \Gamma)$ is a group, then we may assume $\Upsilon = I$ in the statement of Corollary 3.3.

Proof. This follows directly from Theorem 3.1, Theorem 2.5, and Proposition 2.8. \square

To use Corollary 3.3 to obtain isospectral deformations that are not Gordon–Wilson, we exhibit a strictly nonsingular, three-step nilpotent Lie group G and a continuous family of automorphisms $\Phi_s \in \text{Aut}(G)$, $\Phi_s \notin \text{GW}(G, g)$ such that $\bar{\Phi}_s \in \text{GW}(\bar{G}, \bar{g})$. For the examples presented here, $\text{AIA}(G; \Gamma) = \text{AIA}(G)$ and $\text{AIA}(\bar{G}; \bar{\Gamma}) = \text{AIA}(\bar{G})$ for any choice of cocompact discrete subgroup Γ of G .

3.4 Remark. The key to the new deformations is close study of the behavior of the orthogonal automorphisms on the quotient nilmanifold. Example I below exhibits a nontrivial isospectral deformation where almost inner automorphisms play no role whatsoever, i.e., $\text{GW}(\bar{G}, \bar{g}; \bar{\Gamma})$ may be replaced by $\text{OA}(\bar{G}, \bar{g})$ in condition (2) of Corollary 3.3. So the isospectral deformation Φ_s projects to a trivial deformation on the quotient nilmanifold, but Φ_s is nontrivial. If we disregard $\text{OA}(\bar{G}, \bar{g})$, however, we reduce to the case of Gordon–Wilson deformations. For if we let Φ_s satisfy the conditions of Corollary 3.3 and let $\bar{\Phi}_s \in \text{AIA}(\bar{G}; \bar{\Gamma})$ for all s , then the elements $\bar{\gamma}$ and $\bar{\Phi}_s(\bar{\gamma})$ are conjugate for all $\bar{\gamma} \in \bar{\Gamma}$. By strict nonsingularity, γ and $\Phi_s(\gamma)$ are necessarily conjugate, and the deformation is Gordon–Wilson.

3.5 Remark. The geometric role of the orthogonal automorphisms is as follows. Fix an orthonormal basis of the k -dimensional center \mathfrak{z} of \mathfrak{g} and extend it to a $(k+n)$ -dimensional orthonormal basis \mathcal{B} of \mathfrak{g} . If Φ_s satisfies the hypotheses of Corollary 3.3, then $\bar{\Phi}_s = \Upsilon_s \circ \Psi_s$, where $\Upsilon_s \in \text{OA}(\bar{G}, \bar{g})$ and $\Psi_s \in \text{AIA}(\bar{G}, \bar{\Gamma})$. View Υ_{s*} as an element of $O(n)$ by considering its matrix representation with respect to \mathcal{B} . Extend Υ_{s*} to $\hat{\Upsilon}_{s*} \in O(n+k)$ by the $k \times k$ identity matrix:

$$\hat{\Upsilon}_{s*} = \begin{pmatrix} \Upsilon_{s*} & 0 \\ 0 & I_k \end{pmatrix}.$$

The elements of the group $O(n+k)$, hence the matrices $\hat{\Upsilon}_{s*}$, act on our metric g by taking the orthonormal basis \mathcal{B} of \mathfrak{g} to $\hat{\Upsilon}_{s*}^{-1}\mathcal{B}$, which determines the same left invariant metric g of G . So with the obvious notation, we have

$$(\Gamma \backslash G, \Phi_s^* g) = (\Gamma \backslash G, \hat{\Psi}_s^* \circ \hat{\Upsilon}_s^* g) = (\Gamma \backslash G, \hat{\Psi}_s^* g).$$

Here $\hat{\Psi}_{s*} = \hat{\Upsilon}_{s*}^{-1} \circ \Phi_s$, a triangular matrix with 1's down the diagonal, so $\hat{\Psi}_{s*} \in \text{SL}(n+k)$. Note that if \mathfrak{g} is strictly nonsingular, then $\hat{\Psi}_{s*}$ and $\hat{\Upsilon}_{s*}$ are Lie algebra automorphisms if and only if the deformation is Gordon–Wilson.

The matrices $\hat{\Psi}_{s*}$ deform the orthonormal basis, i.e., the metric $\hat{\Psi}_s^* g$ is determined by the orthonormal basis $\hat{\Psi}_{s*}^{-1}\mathcal{B}$. If the deformation is not Gordon–Wilson, then the role of $\hat{\Upsilon}_s$ is to continuously compensate for $\hat{\Psi}_s$ not being an automorphism in a way that does not effect our metric g , allowing us to employ Theorem 3.1. On the other hand, assume that we know $(\Gamma \backslash G, \hat{\Psi}_s^* g)$ is an isospectral deformation and that the $\hat{\Psi}_s$ are not Lie algebra isomorphisms. We might errantly conclude that the manifolds are not locally isometric, i.e., the fact that the manifolds $(\Gamma \backslash G, \hat{\Psi}_s^* g) = (\Phi_s(\Gamma) \backslash G, g)$ are covered by (G, g) is “hidden” by the orthogonal matrices $\hat{\Upsilon}_{s*}$.

We now compute conditions for a Lie algebra automorphism of $\bar{\mathfrak{g}}$ to extend to a Lie group automorphism of the strictly nonsingular Lie algebra \mathfrak{g} .

For the remainder, we assume the center \mathfrak{z} of \mathfrak{g} is one-dimensional. Let G be the strictly nonsingular, nilpotent Lie group with Lie algebra \mathfrak{g} and left invariant metric g . Let \mathfrak{N} be the orthogonal complement of \mathfrak{z} in \mathfrak{g} , so $\mathfrak{g} = \mathfrak{N} \oplus \mathfrak{z}$. Define a closed, nondegenerate two-form ω on \mathfrak{N} by

$$(\star) \quad [X, Y] = ([X, Y]_{\mathfrak{N}}, \omega(X, Y)\mathcal{Z})$$

for all $X, Y \in \mathfrak{g}$. Here $[X, Y]_{\mathfrak{N}}$ denotes the orthogonal projection of $[X, Y]$ onto \mathfrak{N} , and \mathcal{Z} is a unit vector in \mathfrak{z} . The condition that ω is closed is equivalent to the Jacobi equations, and the condition that ω is nondegenerate is equivalent to strict nonsingularity. Note that the vector space \mathfrak{N} with Lie bracket defined by $[\cdot, \cdot]_{\mathfrak{N}}$ is Lie algebra isomorphic to $\bar{\mathfrak{g}}$.

3.7 Notation. Conversely, let (G, g) be a k -step nilpotent Lie group. Assume there exists a closed, nondegenerate two-form ω on \mathfrak{g} . The **central extension of \mathfrak{g} by ω** is the strictly nonsingular, $(k+1)$ -step nilpotent Lie algebra $\hat{\mathfrak{g}}$, where $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}\mathcal{Z}$. The Lie bracket on $\hat{\mathfrak{g}}$ is defined by (\star) , that is, $[(X, Z), (X', Z')]_{\hat{\mathfrak{g}}} = ([X, X']_{\mathfrak{g}}, \omega(X, X')\mathcal{Z})$, for all $(X, Z), (X', Z')$ in $\hat{\mathfrak{g}}$. For $X \in \mathfrak{g}$, we denote by \hat{X} the element $(X, 0)$ in $\hat{\mathfrak{g}}$. We denote by \hat{G} the simply connected nilpotent Lie group with Lie algebra $\hat{\mathfrak{g}}$. Let g be a left invariant metric on G . We extend g to a left invariant metric \hat{g} on \hat{G} by requiring that \mathfrak{g} and \mathcal{Z} be orthogonal and that \mathcal{Z} be a unit vector. Note that the quotient Lie group of (\hat{G}, \hat{g}) is isometric to the Lie group (G, g) .

3.8 Proposition. *Let \mathfrak{g} be a nilpotent Lie algebra, and let ω be a closed, nondegenerate two-form on \mathfrak{g} . Let $\hat{\mathfrak{g}}$ be the central extension of \mathfrak{g} by ω .*

- (1) *A derivation D on \mathfrak{g} extends to a derivation \hat{D} on $\hat{\mathfrak{g}}$ if and only if there exists $\eta \in \mathfrak{g}^*$ such that $D^*\omega = d\eta$, i.e., $D^*\omega$ is exact.*
- (2) *A skew-symmetric derivation S on \mathfrak{g} extends to a skew-symmetric derivation \hat{S} on $\hat{\mathfrak{g}}$ if and only if $S^*\omega = 0$.*

3.9 Notation. If $D^*\omega = d\eta$ with $\eta \in \mathfrak{g}^*$, we denote by \hat{D} the derivation of $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}\mathcal{Z}$ defined by $\hat{D}(\hat{X}) = (D(X), -\eta(X)\mathcal{Z})$ for all $X \in \mathfrak{g}$, and $\hat{D}(\mathcal{Z}) = 0$.

Proof of 3.8. A straightforward calculation using the definition of a derivation shows that D extends to a derivation \hat{D} on $\hat{\mathfrak{g}}$ if and only if \hat{D} is as in (3.9), and (1) follows. The proof of (2) uses (1) and the definition of \hat{g} . \square

3.10 Corollary. *Let G be a strictly nonsingular nilpotent Lie group with Lie algebra \mathfrak{g} and left invariant metric g . Assume there exists a closed, nondegenerate two-form ω on \mathfrak{g} and a derivation $D \in \text{Der}(\mathfrak{g})$ such that*

- (1) *$D = S + A$ where $S \in \text{OD}(\mathfrak{g}, g)$ and $A \in \text{AID}(\mathfrak{g})$, and*
- (2) *$D^*\omega$ is exact.*

Denote by $\hat{D} \in \text{Der}(\hat{\mathfrak{g}})$ the extension of D to $\hat{\mathfrak{g}}$. Define the automorphisms $\Phi_s \in \text{Aut}(\hat{G})$ on the Lie algebra level by $\Phi_{s} = \exp(s\hat{D})$. Let $\hat{\Gamma}$ be any cocompact, discrete subgroup of \hat{G} . Then $(\hat{\Gamma} \backslash \hat{G}, \Phi_s^*\hat{g})$ is an isospectral deformation. The deformation is trivial if and only if $S^*\omega = 0$ and $A \in \text{ID}(\mathfrak{g})$. The deformation is Gordon–Wilson if and only if $S^*\omega = 0$.*

Proof of 3.10. Follows from Corollary 3.3, Proposition 3.8, and Remark 2.7. Note that by strict nonsingularity, if $\Psi \in \text{Aut}(G)$ and $\bar{\Psi} \in \text{AIA}(\bar{G})$, then $\Psi \in \text{AIA}(G)$. \square

3.11 Remark. With appropriate additional assumptions, one may extend the above Corollary to include $\text{AID}(\mathfrak{g}; \Gamma)$. The statement becomes somewhat cumbersome, however, and so is not included here.

Section 4: Examples.

Consider the strictly nonsingular, two-step nilpotent Lie algebra \mathfrak{g} with orthonormal basis $\{X_1, X_2, X_3, X_4, Z_1, Z_2\}$ and Lie bracket given by

$$\begin{aligned} [X_1, X_2] &= [X_3, X_4] = Z_1, \\ [X_1, X_3] &= [X_4, X_2] = Z_2, \end{aligned}$$

and all other basis brackets zero. Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \zeta_1, \zeta_2\}$ be the dual basis of \mathfrak{g}^* , and let G be the simply connected Lie group with Lie algebra \mathfrak{g} .

Example I.

Let

$$\omega = \alpha_1 \wedge \alpha_2 + \alpha_2 \wedge \alpha_3 + \alpha_3 \wedge \alpha_4 + \alpha_1 \wedge \zeta_1 - \alpha_4 \wedge \zeta_2.$$

The closed, nondegenerate two-form ω induces a strictly nonsingular, three-step nilpotent Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}\mathcal{Z}$ with Lie bracket and left invariant metric as defined in (3.7). So $\hat{\mathfrak{g}}$ is the 7-dimensional Lie algebra with orthonormal basis $\{\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{Z}_1, \hat{Z}_2, \mathcal{Z}\}$ and Lie bracket

$$\begin{aligned} [\hat{X}_1, \hat{X}_2] &= [\hat{X}_3, \hat{X}_4] = \hat{Z}_1 + \mathcal{Z}, \\ [\hat{X}_1, \hat{X}_3] &= [\hat{X}_4, \hat{X}_2] = \hat{Z}_2, \\ [\hat{X}_2, \hat{X}_3] &= [\hat{X}_1, \hat{Z}_1] = [\hat{Z}_2, \hat{X}_4] = \mathcal{Z}, \end{aligned}$$

and all other basis brackets zero.

One easily checks that the skew-symmetric derivation $S : \mathfrak{g} \rightarrow \mathfrak{g}$ defined by

$$X_1 \rightarrow -X_4; \quad X_2 \rightarrow 2X_3; \quad X_3 \rightarrow -2X_2; \quad X_4 \rightarrow X_1; \quad Z_1 \rightarrow Z_2; \quad Z_2 \rightarrow -Z_1$$

satisfies $S^*\omega = d\zeta_2$, so S extends to the derivation $\hat{S} : \hat{\mathfrak{g}} \rightarrow \hat{\mathfrak{g}}$ given by $\hat{S}(\hat{X}) = (S(X), -\zeta_2(X)\mathcal{Z})$, $\hat{S}(\mathcal{Z}) = 0$. That is, \hat{S} sends

$$\hat{X}_1 \rightarrow -\hat{X}_4; \quad \hat{X}_2 \rightarrow 2\hat{X}_3; \quad \hat{X}_3 \rightarrow -2\hat{X}_2; \quad \hat{X}_4 \rightarrow \hat{X}_1; \quad \hat{Z}_1 \rightarrow \hat{Z}_2; \quad \hat{Z}_2 \rightarrow -\hat{Z}_1 - \mathcal{Z}; \quad \mathcal{Z} \rightarrow 0.$$

Let \hat{G} denote the simply connected, strictly nonsingular nilpotent Lie group with Lie algebra $\hat{\mathfrak{g}}$. Define a continuous family Φ_s of Lie group automorphisms on \hat{G} given at the Lie algebra level by $\Phi_{s*} = \exp(s\hat{S})$. Using any symbolic algebra package, one computes that the automorphism Φ_{s*} sends

$$\begin{aligned} \hat{X}_1 &\rightarrow \cos(s)\hat{X}_1 - \sin(s)\hat{X}_4, \\ \hat{X}_2 &\rightarrow \cos(2s)\hat{X}_2 + \sin(2s)\hat{X}_3, \\ \hat{X}_3 &\rightarrow -\sin(2s)\hat{X}_2 + \cos(2s)\hat{X}_3, \\ \hat{X}_4 &\rightarrow \sin(s)\hat{X}_1 + \cos(s)\hat{X}_4, \\ \hat{Z}_1 &\rightarrow \cos(s)\hat{Z}_1 + \sin(s)\hat{Z}_2 - (1 - \cos(s))\mathcal{Z}, \\ \hat{Z}_2 &\rightarrow -\sin(s)\hat{Z}_1 + \cos(s)\hat{Z}_2 - \sin(s)\mathcal{Z}, \\ \mathcal{Z} &\rightarrow \mathcal{Z}. \end{aligned}$$

Let $\hat{\Gamma}$ be any cocompact, discrete subgroup of \hat{G} and let $\hat{\Gamma}_s = \Phi_s(\hat{\Gamma})$. Then by Corollary 3.10, $(\hat{\Gamma}_s \backslash \hat{G}, \hat{g}) = (\hat{\Gamma} \backslash \hat{G}, \Phi_s^* \hat{g})$ is an isospectral deformation that is not Gordon–Wilson. Theorem 6.1 below shows that $1\text{-spec}(\Phi_s(\hat{\Gamma}) \backslash \hat{G}, \hat{g})$ is nonconstant, providing an alternate proof that the deformation is not through almost inner automorphisms.

By factoring out the orthogonal matrices as in (3.5), the left invariant metric $\Phi_s^* \hat{g}$ is determined by the orthonormal basis $\{\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{Z}_1 + (1 - \cos(s))\mathcal{Z}, \hat{Z}_2 + \sin(s)\mathcal{Z}, \mathcal{Z}\}$ of $\hat{\mathfrak{g}}$.

Example II.

Let

$$\omega = \alpha_2 \wedge \alpha_3 + \alpha_2 \wedge \alpha_4 + \alpha_1 \wedge \zeta_1 - \alpha_4 \wedge \zeta_2.$$

The closed, nondegenerate two-form ω induces a strictly nonsingular, three-step nilpotent Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}\mathcal{Z}$, with Lie bracket and left invariant metric as defined in (3.7). So \mathfrak{g} is the 7-dimensional Lie algebra with orthonormal basis $\{\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{Z}_1, \hat{Z}_2, \mathcal{Z}\}$, and Lie bracket

$$\begin{aligned} [\hat{X}_1, \hat{X}_2] &= [\hat{X}_3, \hat{X}_4] = \hat{Z}_1, \\ [\hat{X}_1, \hat{X}_3] &= \hat{Z}_2, \quad [\hat{X}_4, \hat{X}_2] = \hat{Z}_2 - \mathcal{Z}, \\ [\hat{X}_2, \hat{X}_3] &= [\hat{X}_1, \hat{Z}_1] = [\hat{Z}_2, \hat{X}_4] = \mathcal{Z}, \end{aligned}$$

and all other basis brackets zero.

One easily checks that the derivations $S : \mathfrak{g} \rightarrow \mathfrak{g}$ defined by

$$X_1 \rightarrow -X_4; \quad X_2 \rightarrow 2X_3; \quad X_3 \rightarrow -2X_2; \quad X_4 \rightarrow X_1; \quad Z_1 \rightarrow Z_2; \quad Z_2 \rightarrow -Z_1;$$

and $A : \mathfrak{g} \rightarrow \mathfrak{g}$ defined by

$$X_1, X_4, Z_1, Z_2 \rightarrow 0; \quad X_2 \rightarrow -Z_1; \quad X_3 \rightarrow 2Z_2$$

satisfy $S^* \omega = \alpha_1 \wedge \alpha_2 - 2\alpha_3 \wedge \alpha_4$, and $A^* \omega = -S^* \omega$. Note that $S^* \omega$ is not in $\text{span}_{\mathbb{R}}\{d\zeta_1, d\zeta_2, d\xi\}$, hence not exact. So neither S nor A will extend to a derivation on $\hat{\mathfrak{g}}$. However, the sum $D = A + S$ is exact as $D^* \omega = 0$, so D extends to the derivation $\hat{D} : \hat{\mathfrak{g}} \rightarrow \hat{\mathfrak{g}}$ given by $\hat{D}(\hat{X}) = (S(X) + A(X), 0)$, $\hat{D}(\mathcal{Z}) = 0$. That is, \hat{D} sends

$$\begin{aligned} \hat{X}_1 &\rightarrow -\hat{X}_4; \quad \hat{X}_2 \rightarrow 2\hat{X}_3 - \hat{Z}_1; \quad \hat{X}_3 \rightarrow -2\hat{X}_2 + 2\hat{Z}_2; \quad \hat{X}_4 \rightarrow \hat{X}_1; \\ \hat{Z}_1 &\rightarrow \hat{Z}_2; \quad \hat{Z}_2 \rightarrow -\hat{Z}_1; \quad \mathcal{Z} \rightarrow 0. \end{aligned}$$

Let \hat{G} denote the simply connected, strictly nonsingular nilpotent Lie group with Lie algebra $\hat{\mathfrak{g}}$. Define a continuous family Φ_s of Lie group automorphisms

of \hat{G} defined on the Lie algebra level by $\Phi_{s*} = \exp(s\hat{D})$. The automorphism Φ_{s*} sends

$$\begin{aligned}\hat{X}_1 &\rightarrow \cos(s)\hat{X}_1 - \sin(s)\hat{X}_4, \\ \hat{X}_2 &\rightarrow \cos(2s)\hat{X}_2 + \sin(2s)\hat{X}_3 - \sin(s)\hat{Z}_1 + (\cos(s) - \cos(2s))\hat{Z}_2, \\ \hat{X}_3 &\rightarrow -\sin(2s)\hat{X}_2 + \cos(2s)\hat{X}_3 + \sin(2s)\hat{Z}_2, \\ \hat{X}_4 &\rightarrow \sin(s)\hat{X}_1 + \cos(s)\hat{X}_4, \\ \hat{Z}_1 &\rightarrow \cos(s)\hat{Z}_1 + \sin(s)\hat{Z}_2, \\ \hat{Z}_2 &\rightarrow -\sin(s)\hat{Z}_1 + \cos(s)\hat{Z}_2, \\ \mathcal{Z} &\rightarrow \mathcal{Z}.\end{aligned}$$

Let $\hat{\Gamma}$ be any cocompact, discrete subgroup of \hat{G} . Then $(\Phi_s(\hat{\Gamma}) \backslash \hat{G}, \hat{g}) = (\hat{\Gamma} \backslash \hat{G}, \Phi_s^* \hat{g})$ is an isospectral deformation not arising from almost inner automorphisms. We show in Theorem 6.2 below that $1\text{-spec}(\Phi_s(\hat{\Gamma}) \backslash \hat{G}, \hat{g})$ is nonconstant, providing an alternate proof that the deformation is not Gordon–Wilson.

By factoring out the orthogonal matrices as in (3.5), we see that $\Phi_s^* \hat{g}$ is determined by the orthonormal basis $\{\hat{X}_1, \hat{X}_2 + \sin(s)\cos(2s)\hat{Z}_1 - (1 - \cos(s)\cos(2s))\hat{Z}_2, \hat{X}_3 - \sin(s)\sin(2s)\hat{Z}_1 - \cos(s)\sin(2s)\hat{Z}_2, \hat{X}_4, \hat{Z}_1, \hat{Z}_2, \mathcal{Z}\}$ of $\hat{\mathfrak{g}}$.

Section 5. The Marked Length Spectrum.

In this section we prove the following.

5.1 Theorem. *The nilmanifolds $(\hat{\Gamma}_s \backslash \hat{G}, \hat{g})$ presented in Example I above have the same marked length spectrum for all s . In particular, Φ_s marks the length spectrum from $(\hat{\Gamma}_0 \backslash \hat{G}, \hat{g})$ to $(\hat{\Gamma}_s \backslash \hat{G}, \hat{g})$.*

5.2 Theorem. *The nilmanifolds $(\hat{\Gamma}_s \backslash \hat{G}, \hat{g})$ presented in Example II above have the same marked length spectrum for all s . In particular, Φ_s marks the length spectrum from $(\hat{\Gamma}_0 \backslash \hat{G}, \hat{g})$ to $(\hat{\Gamma}_s \backslash \hat{G}, \hat{g})$.*

5.3 Remark. These are the first continuous families of Riemannian manifolds with the same marked length spectrum but not the same spectrum on 1-forms. Discrete families of manifolds with these properties were presented in [Gt4]. The only other known examples of manifolds that have the same marked length spectrum but not the same 1-form spectrum are the Zoll spheres and standard spheres, which are not even isospectral on functions.

If G is a simply connected Lie group with cocompact, discrete subgroup Γ , then Γ is also the fundamental group of $\Gamma \backslash G$, so free homotopy classes of loops of $\Gamma \backslash G$ correspond to conjugacy classes in Γ . We denote by $[\gamma]_\Gamma$ the conjugacy class in Γ represented by $\gamma \in \Gamma$, that is, $[\gamma]_\Gamma = \{\gamma' \gamma \gamma'^{-1} : \gamma' \in \Gamma\}$. Recall that any isomorphism between fundamental groups induces a correspondence between free homotopy classes of loops.

5.4 Definition. A pair of Riemannian manifolds $(\Gamma_1 \backslash \tilde{M}_1, g_1)$ and $(\Gamma_2 \backslash \tilde{M}_2, g_2)$ have the same **marked length spectrum** if and only if there exists an isomorphism $\Phi : \Gamma_1 \rightarrow \Gamma_2$ between their fundamental groups such that corresponding free homotopy classes contain closed geodesics of the same length. That is, the free homotopy class represented by $[\gamma]_{\Gamma_1}$ contains a closed geodesic of (positive) period λ if and only if the free homotopy class represented by $[\Phi(\gamma)]_{\Gamma_2}$ also contains a closed geodesic of period λ . We say that Φ **marks the length spectrum** from $(\Gamma_1 \backslash \tilde{M}_1, g_1)$ to $(\Gamma_2 \backslash \tilde{M}_2, g_2)$.

We have the following sufficient condition for a pair of nilmanifolds to have the same marked length spectrum.

5.5 Theorem [G1, Theorem 1.8]. *Let G be a simply connected nilpotent Lie group with cocompact, discrete subgroup Γ . Let Φ be a Γ -almost inner automorphism of G . Then for any left invariant metric g of G , Φ marks the length spectrum from $(\Gamma \backslash G, g)$ to $(\Phi(\Gamma) \backslash G, g)$.*

Trivially, if Φ is an orthogonal automorphism, i.e., $\Phi \in \text{OA}(G, g)$, then Φ marks the length spectrum from $(\Gamma \backslash G, g)$ to $(\Phi(\Gamma) \backslash G, g)$. Conversely, on two-step nilmanifolds Eberlein showed the following.

5.6 Theorem [E, Theorem 5.20]. *Let Γ_1, Γ_2 be cocompact, discrete subgroups of simply connected, two-step nilpotent Lie groups G_1, G_2 with left invariant metrics g_1, g_2 respectively. Assume that $(\Gamma_1 \backslash G_1, g_1)$ and $(\Gamma_2 \backslash G_2, g_2)$ have the same marked length spectrum, and let $\Phi : \Gamma_1 \rightarrow \Gamma_2$ be an isomorphism inducing this marking. Then $\Phi = (\Phi_1 \circ \Phi_2)|_{\Gamma_1}$, where Φ_2 is a Γ_1 -almost inner automorphism of G_1 , and Φ_1 is an isomorphism of (G_1, g_1) onto (G_2, g_2) that is also an isometry. Moreover, this factorization is unique. In particular, if $(\Gamma_1 \backslash G_1, g_1)$ and $(\Gamma_2 \backslash G_2, g_2)$ have the same marked length spectrum, they necessarily have the same spectrum of the Laplacian on functions and on p -forms for all p .*

In contrast to (5.6), we have the following necessary and sufficient condition for a large family of pairs of three-step nilmanifolds to have the same marked length spectrum.

5.7 Theorem [Gt4, Theorem 3.3.1]. *Let G be a simply connected, strictly non-singular, three-step nilpotent Lie group with a one-dimensional center. Let Γ_1 and Γ_2 be cocompact, discrete subgroups of G such that $\Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G)$. Let g be any left invariant metric on G . Then $(\Gamma_1 \backslash G, g)$ and $(\Gamma_2 \backslash G, g)$ have the same marked length spectrum if and only if there exists an isomorphism $\Phi : \Gamma_1 \rightarrow \Gamma_2$ such that $\bar{\Phi} : \bar{\Gamma}_1 \rightarrow \bar{\Gamma}_2$ marks the length spectrum between $(\bar{\Gamma}_1 \backslash \bar{G}, \bar{g})$ and $(\bar{\Gamma}_2 \backslash \bar{G}, \bar{g})$.*

Proof of 5.1 and 5.2. The automorphisms Φ_s presented in Example I and Example II have the property that $\bar{\Phi}_s$ is the composition of an almost inner automorphism and an orthogonal automorphism, which by (5.5) necessarily

mark the length spectrum from $(\Gamma_0 \backslash G, g)$ to $(\Gamma_s \backslash G, g)$. The result now follows from (5.7). \square

We also showed in [Gt4] that for a large class of three-step nilmanifolds, which include Example I and Example II, if a family of nilmanifolds in this class have the same marked length spectrum, they necessarily share the same Laplace spectrum on functions, providing an alternate proof that Example I and Example II are isospectral deformations.

5.8 Theorem [Gt4, Theorem 3.3.2]. *Let G be a simply connected, strictly nonsingular, three-step nilpotent Lie group. Let Γ_1 and Γ_2 be cocompact, discrete subgroups of G such that $\Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G)$. If $(\Gamma_1 \backslash G, g)$ and $(\Gamma_2 \backslash G, g)$ have the same marked length spectrum, then $(\Gamma_1 \backslash G, g)$ and $(\Gamma_2 \backslash G, g)$ are isospectral on functions.*

Section 6: Comparing the 1-form spectrum of nilmanifolds.

In this section, we prove the following.

6.1 Theorem. *The nilmanifolds $(\hat{\Gamma}_s \backslash \hat{G}, \hat{g})$ of Example I are not isospectral on 1-forms.*

6.2 Theorem. *The nilmanifolds $(\hat{\Gamma}_s \backslash \hat{G}, \hat{g})$ of Example II are not isospectral on 1-forms.*

Proof of Theorem 6.1.

Let \hat{G} , $\hat{\mathfrak{g}}$, \hat{g} , and Φ_s be as in Example I, Section 4. Let $\hat{\Gamma}$ be an arbitrary cocompact, discrete subgroup of \hat{G} , and let $\hat{\Gamma}_s = \Phi_s(\hat{\Gamma})$. We calculate a monic polynomial $P_s(x)$ of degree 7 whose roots must appear in $1\text{-spec}(\hat{\Gamma}_s \backslash \hat{G}, \hat{g})$. We show that the coefficients of $P_s(x)$, which must change continuously with s , are nonconstant. The result follows.

We refer the reader to [Gt3, Appendix] for definitions, notation, and an exposition of the method used below to demonstrate the inequality of the 1-form spectrum. We include only those details needed for verification.

The main tool we use is representation theory. Irreducible representations of nilpotent Lie groups are well understood due to Kirillov theory. In particular, the irreducible representations of a nilpotent Lie group G correspond to the coadjoint orbits of the Lie algebra \mathfrak{g} of G . For $\tau \in \mathfrak{g}^*$, we denote by π_τ the corresponding irreducible representation of G . A good reference for representations of nilpotent Lie groups is [CG].

We use only the characters of \hat{G} , i.e., the (complex) one-dimensional irreducible representations to demonstrate the inequality of the 1-form spectrum. The dimension of the representation space \mathcal{H}_τ of π_τ corresponds to the dimension of the coadjoint orbit of τ . In particular, the one-dimensional representations of G correspond to those elements of \mathfrak{g}^* that are zero on the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ of \mathfrak{g} .

Consider $\rho_{\hat{\Gamma}_s}$, the quasi-regular representation of \hat{G} on $L^2(\hat{\Gamma}_s \backslash \hat{G})$. For $f \in L^2(\hat{\Gamma}_s \backslash \hat{G})$, and $x \in \hat{G}$, $\rho_{\hat{\Gamma}_s}(x)f = f \circ R_x$, where R denotes the right action of \hat{G} on $\hat{\Gamma}_s \backslash \hat{G}$. It is known that $L^2(\hat{\Gamma}_s \backslash \hat{G})$ is completely reducible as

$$L^2(\hat{\Gamma}_s \backslash \hat{G}) \cong \bigoplus_{\tau \in \mathcal{T}_s} m_s(\tau) \mathcal{H}_\tau.$$

Here $m_s(\tau)$ is the multiplicity of π_τ in $\rho_{\hat{\Gamma}_s}$, and \mathcal{T}_s is a subset of $\hat{\mathfrak{g}}^*$ containing at most one element of each coadjoint orbit of $\hat{\mathfrak{g}}^*$.

Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \zeta_1, \zeta_2, \xi\}$ be the dual basis to the orthonormal basis $\{\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{Z}_1, \hat{Z}_2, \mathcal{Z}\}$ of $\hat{\mathfrak{g}}$. Denote by \mathfrak{C} the subspace $\mathfrak{C} = \text{span}_{\mathbb{R}}\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ of $\hat{\mathfrak{g}}^*$. Note that $\mathfrak{C} = \{\tau \in \hat{\mathfrak{g}}^* : \tau([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]) \equiv 0\}$. Thus the characters of \hat{G} appearing in $L^2(\hat{\Gamma}_s \backslash \hat{G})$ correspond to $\mathcal{T}_s \cap \mathfrak{C}$.

Let $\tau \in \mathfrak{C}$. We now calculate Δ acting on $\mathcal{H}_\tau \otimes \Lambda^1(\hat{\mathfrak{g}}^*)$, an invariant subspace of Δ . Note that if $\mathbb{C}\Lambda^1(\hat{\mathfrak{g}}^*)$ denotes $\Lambda^1(\hat{\mathfrak{g}}^*)$ with complex coefficients, then $\mathcal{H}_\tau \otimes \Lambda^1(\hat{\mathfrak{g}}^*) = F_\tau \otimes \mathbb{C}\Lambda^1(\hat{\mathfrak{g}}^*)$, where $F_\tau(x) = e^{(2\pi i \tau(\log(x)))}$.

As $\hat{Z}_1 F_\tau = \hat{Z}_2 F_\tau = \mathcal{Z} F_\tau = 0$, we know $\Delta F_\tau = 4\pi^2 \|\tau\|^2 F_\tau$. One also computes the following.

$$\Delta\alpha_1 = \Delta\alpha_2 = \Delta\alpha_3 = \Delta\alpha_4 = 0, \quad \Delta\zeta_1 = 2\zeta_1 + 2\xi, \quad \Delta\zeta_2 = 2\zeta_2, \quad \Delta\xi = 2\zeta_1 + 5\xi$$

$\nabla_U \mu$	α_1	α_2	α_3	α_4	ζ_1	ζ_2	ξ
X_1	0	$\frac{1}{2}(\zeta_1 + \xi)$	$\frac{1}{2}\zeta_2$	0	$\frac{1}{2}(-\alpha_2 + \xi)$	$-\frac{1}{2}\alpha_3$	$-\frac{1}{2}(\alpha_2 + \zeta_1)$
X_2	$-\frac{1}{2}(\zeta_1 + \xi)$	0	$\frac{1}{2}\xi$	$-\frac{1}{2}\zeta_2$	$\frac{1}{2}\alpha_1$	$\frac{1}{2}\alpha_4$	$\frac{1}{2}(\alpha_1 - \alpha_3)$
X_3	$-\frac{1}{2}\zeta_2$	$-\frac{1}{2}\xi$	0	$\frac{1}{2}(\zeta_1 + \xi)$	$-\frac{1}{2}\alpha_4$	$\frac{1}{2}\alpha_1$	$\frac{1}{2}(\alpha_2 - \alpha_4)$
X_4	0	$\frac{1}{2}\zeta_2$	$-\frac{1}{2}(\zeta_1 + \xi)$	0	$\frac{1}{2}\alpha_3$	$-\frac{1}{2}(\alpha_2 + \xi)$	$\frac{1}{2}(\alpha_3 + \zeta_2)$

Write $\tau = A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 + A_4\alpha_4$ where $A_i \in \mathbb{R}$. Using $\Delta(f\tau) = (\Delta f)\tau + f(\Delta\tau) - 2\nabla_{\text{grad}f}\tau$ and the above information, a straightforward calculation shows that if we set $E_\tau =$

$$\begin{pmatrix} 4\pi^2 \|\tau\|^2 & 0 & 0 & 0 & 2\pi i A_2 & 2\pi i A_3 & 2\pi i A_2 \\ 0 & 4\pi^2 \|\tau\|^2 & 0 & 0 & -2\pi i A_1 & -2\pi i A_4 & 2\pi i (A_3 - A_1) \\ 0 & 0 & 4\pi^2 \|\tau\|^2 & 0 & 2\pi i A_4 & -2\pi i A_1 & 2\pi i (A_4 - A_2) \\ 0 & 0 & 0 & 4\pi^2 \|\tau\|^2 & -2\pi i A_3 & 2\pi i A_2 & -2\pi i A_3 \\ -2\pi i A_2 & 2\pi i A_1 & -2\pi i A_4 & 2\pi i A_3 & 4\pi^2 \|\tau\|^2 + 2 & 0 & -2\pi i A_1 + 2 \\ -2\pi i A_3 & 2\pi i A_4 & 2\pi i A_1 & -2\pi i A_2 & 0 & 4\pi^2 \|\tau\|^2 + 2 & 2\pi i A_4 \\ -2\pi i A_2 & 2\pi i (A_1 - A_3) & 2\pi i (A_2 - A_4) & 2\pi i A_3 & 2\pi i A_1 + 2 & -2\pi i A_4 & 4\pi^2 \|\tau\|^2 + 5 \end{pmatrix}$$

then there exists $\mu \in \mathbb{C}\Lambda^1(\hat{\mathfrak{g}}^*)$, such that $\Delta(F_\tau \otimes \mu) = \lambda(F_\tau \otimes \mu)$ if and only if λ is an eigenvalue of the matrix E_τ . That is, for every $\tau \in \mathcal{T}_s \cap \mathfrak{C}$, the roots of the characteristic polynomial of E_τ are contained in $1\text{-spec}(\Gamma_s \backslash G, g)$. One easily computes that $\tau \in \mathcal{T}_s \cap \mathfrak{C}$ if and only if $\tau(\log \hat{\Gamma}_s) \subset \mathbb{Z}$.

Now $\log \hat{\Gamma}_s$ generates a lattice of full rank in $\hat{\mathfrak{g}}$, so we can find $V_1^s, V_2^s, V_3^s, V_4^s$, elements of $\text{span}_{\mathbb{R}}\{\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4\}$, with the property that $\tau(\log \hat{\Gamma}_s) \subset \mathbb{Z}$ if and only if $\tau(V_i^s) \subset \mathbb{Z}$, $i = 1, 2, 3, 4$. Let $L_s = \text{span}_{\mathbb{Z}}\{V_1^s, V_2^s, V_3^s, V_4^s\}$. Note that L_s is a lattice of full rank in $\text{span}_{\mathbb{R}}\{\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4\}$. Let L_s^* be the dual lattice of L_s in \mathfrak{C} , so L_s^* is a lattice of full rank in \mathfrak{C} , and

$$\tau \in \mathcal{T}_s \cap \mathfrak{C} \text{ if and only if } \tau \in L_s^*.$$

A straightforward computation shows that if $\tau = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4$ is any element of $L_0^* \subset \mathcal{T}_0 \cap \mathfrak{C}$ then $\tau_s = \tau \circ \Phi_{s*}^{-1}$ is an element of $L_s^* \subset \mathcal{T}_s \cap \mathfrak{C}$. That is, $\tau_s = A_1(s)\alpha_1 + A_2(s)\alpha_2 + A_3(s)\alpha_3 + A_4(s)\alpha_4$ is an element of L_s^* , where

$$\begin{aligned} A_1(s) &= a_1 \cos(s) + a_4 \sin(s) \\ A_2(s) &= a_2 \cos(2s) - a_3 \sin(2s) \\ A_3(s) &= a_2 \sin(2s) + a_3 \cos(2s) \\ A_4(s) &= -a_1 \sin(s) + a_4 \cos(s). \end{aligned}$$

Note that $\|\tau_s\|^2 = a_1^2 + a_2^2 + a_3^2 + a_4^2$ is independent of s . Using any symbolic computation package, one easily checks that the characteristic polynomial of E_{τ_s} , denoted $P_s(x)$, is a monic polynomial of degree 7, and the coefficients of the monic polynomial $P_s(x)$ are independent of s if and only if $a_1a_2 = a_3a_4$. As L_0^* is full lattice in \mathfrak{C} , there exists at least one $\tau \in L_0^*$ satisfying $a_1a_2 \neq a_3a_4$, and we are done. \square

6.3 Remark. Alternatively, one may perform the same computation by viewing the manifolds of Example I as $(\hat{\Gamma} \backslash \hat{G}, \Phi_s^* \hat{g})$. This is simplified by factoring out an orthogonal matrix for each s , as in (3.5), obtaining the orthonormal basis $\{\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{Z}_1 + (1 - \cos(s))\mathcal{Z}, \hat{Z}_2 + \sin(s)\mathcal{Z}, \mathcal{Z}\}$ of $\hat{\mathfrak{g}}$. Here the functional τ remains fixed, but the values for $\Delta\alpha_i$, $\Delta\zeta_i$ and $\nabla_U \mu$ listed above now vary with s . One obtains a different expression for E_τ depending on s , but the characteristic polynomial $P_s(x)$ is equal to that obtained above.

Proof of Theorem 6.2.

Let \hat{G} , $\hat{\mathfrak{g}}$, \hat{g} , and Φ_s be as in Example II, Section 4. Let $\hat{\Gamma}$ be an arbitrary co-compact, discrete subgroup of \hat{G} , and let $\hat{\Gamma}_s = \Phi_s(\hat{\Gamma})$. Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \zeta_1, \zeta_2, \xi\}$ be the dual basis to the orthonormal basis $\{\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{Z}_1, \hat{Z}_2, \mathcal{Z}\}$ of $\hat{\mathfrak{g}}$.

The proof here is identical to that of Example I, with the following modifications.

$$\Delta\alpha_1 = \Delta\alpha_2 = \Delta\alpha_3 = \Delta\alpha_4 = 0, \quad \Delta\zeta_1 = 2\zeta_1, \quad \Delta\zeta_2 = 2\zeta_2 - \xi, \quad \Delta\xi = -\zeta_1 + 4\xi.$$

$\nabla_U \mu$	α_1	α_2	α_3	α_4	ζ_1	ζ_2	ξ
X_1	0	$\frac{1}{2}\zeta_1$	$\frac{1}{2}\zeta_2$	0	$\frac{1}{2}(-\alpha_2 + \xi)$	$-\frac{1}{2}\alpha_3$	$-\frac{1}{2}\zeta_1$
X_2	$-\frac{1}{2}\zeta_1$	0	$\frac{1}{2}\xi$	$\frac{1}{2}(-\zeta_2 + \xi)$	$\frac{1}{2}\alpha_1$	$\frac{1}{2}\alpha_4$	$-\frac{1}{2}(\alpha_3 + \alpha_4)$
X_3	$-\frac{1}{2}\zeta_2$	$-\frac{1}{2}\xi$	0	$\frac{1}{2}\zeta_1$	$-\frac{1}{2}\alpha_4$	$\frac{1}{2}\alpha_1$	$\frac{1}{2}\alpha_2$
X_4	0	$\frac{1}{2}(\zeta_2 - \xi)$	$-\frac{1}{2}\zeta_1$	0	$\frac{1}{2}\alpha_3$	$-\frac{1}{2}(\alpha_2 + \xi)$	$\frac{1}{2}(\alpha_2 + \zeta_2)$

Set $E_\tau =$

$$\begin{pmatrix} 4\pi^2\|\tau\|^2 & 0 & 0 & 0 & 2\pi i A_2 & 2\pi i A_3 & 0 \\ 0 & 4\pi^2\|\tau\|^2 & 0 & 0 & -2\pi i A_1 & -2\pi i A_4 & 2\pi i(A_3 + A_4) \\ 0 & 0 & 4\pi^2\|\tau\|^2 & 0 & 2\pi i A_4 & -2\pi i A_1 & -2\pi i A_2 \\ 0 & 0 & 0 & 4\pi^2\|\tau\|^2 & -2\pi i A_3 & 2\pi i A_2 & -2\pi i A_2 \\ -2\pi i A_2 & 2\pi i A_1 & -2\pi i A_4 & 2\pi i A_3 & 4\pi^2\|\tau\|^2 + 2 & 0 & -2\pi i A_1 \\ -2\pi i A_3 & 2\pi i A_4 & 2\pi i A_1 & -2\pi i A_2 & 0 & 4\pi^2\|\tau\|^2 + 2 & 2\pi i A_4 - 1 \\ 0 & -2\pi i(A_3 + A_4) & 2\pi i A_2 & 2\pi i A_2 & 2\pi i A_1 & -2\pi i A_4 - 1 & 4\pi^2\|\tau\|^2 + 4 \end{pmatrix}$$

We may use the same values for $A_i(s)$, $i = 1, 2, 3, 4$. \square

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